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Bounded Sobolev norms for linear Schrödinger equations under resonant perturbations

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Abstract

We prove that the Sobolev norms for a 1-D periodic Schrödinger equation remain bounded under small resonant perturbations.

Keywords: Sobolev norms; Floquet Hamiltonian

Contents

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1. Introduction and statement of the theorems

We consider time periodic Schrödinger equations with periodic boundary conditions of the form

$$i\frac{\partial}{\partial t}u = -\Delta u + V(x,t)u, \tag{1.1}$$

where $x \in \mathbb{T}$, $t \in \mathbb{R}$, and V is a real analytic potential periodic in x and t. The spectrum of the Laplacian: $\sigma(-\Delta) = \{j^2, j \in \mathbb{Z}\}$. We are interested in *resonant* perturbations, i.e., when the frequency ω of the time periodic potential V is an integer, $\omega \in \mathbb{Z} \setminus \{0\}$.

Assume u is a solution, the L^2 norm is conserved by the Schrödinger flow map:

$$\|u(t)\|_{L^2(\mathbb{T})} = \|u(0)\|_{L^2(\mathbb{T})}, \tag{1.2}$$

for all t, and if $u(0) \in H^s(\mathbb{T})$ $(s > 0)$, then $u(t) \in H^s(\mathbb{T})$ for all t. Unlike the time independent Schrödinger equations, in general, the H^s norms of solutions to time dependent equations as in (1.1) can grow in time. Here we are concerned with the bounds on the Sobolev norms: $\|u(t)\|_{H^s(\mathbb{T})}$ as $t \to \infty$ when V is resonant. This is generally speaking a more "dangerous" case, where there is possible growth of H^s norms.

We note that for a linear equation of the form (1.1), if one assumes V is smooth in x and t (not necessarily periodic), one has the a priori bound (cf. [2,3]):

$$\|u(t)\|_{H^s} \leqslant C_s\left(1+|t|^s\right)\|u(0)\|_{H^s}, \tag{1.3}$$

where $u(t)$ is the solution to (1.1) with initial condition $u(0)$.

Under a natural spectral condition (cf. (H1) of Theorem 0 in Section 5), we prove that the H^s norms of solutions to (1.1) remain bounded for all $s > 0$,

$$\|u(t)\|_{H^s} \leqslant C_s\|u(0)\|_{H^s}, \tag{1.4}$$

for all t. We show that this spectral condition (H1) is verified for small potentials V.

Previously in [1], it was shown that for time quasi-periodic potentials with Diophantine frequencies (hence non-resonant)

$$\|u(t)\|_{H^s} \leqslant C_s\left(\log\left(1+|t|\right)\right)^{C_s}\|u(0)\|_{H^s}, \tag{1.5}$$

for the corresponding solutions to (1.1). (1.5) holds in 1d and 2d when the time quasi-periodic potential V is small. In the periodic case, (1.5) was observed by T. Spencer [10], with no assumptions on the frequency ω. The present paper constructs an explicit example where there is *no* growth of Sobolev norms. It is partially motivated by results in [1–3,10].

In a companion paper [11], using related constructions, we show that for a general bounded, time dependent potential $V(x,t)$, $x \in \mathbb{T}$ and $t \in \mathbb{R}$, which is analytic and periodic in x, smooth in t (with no further specifications on the time dependence), the growth of Sobolev norms is at most logarithmic in t. Previously, Bourgain [2,3] showed that the growth of Sobolev norms is at most polynomial in t: t^ϵ (for any $\epsilon > 0$) for $V(x,t)$ bounded, periodic in x, smooth in x and t.

The Floquet Hamiltonian. When V is periodic in time, it is well known from [4,8,12] that properties of the solutions to (1.1) can be deduced from the spectral properties of the corresponding Floquet Hamiltonian:

$$H = -i\frac{\partial}{\partial t} - \Delta + V \tag{1.6}$$

on $L^2(\mathbb{T}_x \times \mathbb{T}_{t,\omega})$, where $\mathbb{T}_x = [0, 2\pi)$ with periodic boundary conditions and $T_{t,\omega} = [0, 2\pi/\omega)$ with periodic boundary conditions.

By Fourier series, H is unitarily equivalent to

$$\hat{H} = \operatorname{diag}\bigl(n\omega + j^2\bigr) + \hat{V} * \quad \text{on } \ell^2\bigl(\mathbb{Z}^2\bigr), \tag{1.7}$$

where $\hat{V}(j,n)$ are the Fourier coefficients of V:

$$V(x,t) = \sum_{(j,n)\in\mathbb{Z}^2} \hat{V}(j,n)e^{i(jx+n\omega t)}, \tag{1.8}$$

and $\hat{V}*$ denotes convolution:

$$[\hat{V} * u](j,n) = \sum_{(j',n')} \hat{V}(j-j', n-n')u(j',n'). \tag{1.9}$$

In this paper, for simplicity, we take

$$V(x,t) = 2\cos x \cos t, \tag{1.10}$$

instead of a more general analytic periodic potential. The frequency $\omega = 1$ here. The method here applies in the general case. (The term $2\hat{V}(0,1)\cos t$ can be eliminated by replacing u by $ue^{2i\sin t\hat{V}(0,1)}$. This elimination procedure clearly holds more generally for potentials which only depend on t.) The Floquet Hamiltonian is then

$$\hat{H} = \operatorname{diag}\bigl(n + j^2\bigr) + \tilde{\Delta}, \tag{1.11}$$

where

$$\tilde{\Delta}(j,n;j',n') = \begin{cases} 1, & |j-j'|_{\ell^1} = 1 \text{ and } |n-n'|_{\ell^1} = 1, \\ 0, & \text{otherwise.} \end{cases} \tag{1.12}$$

Writing H for $\hat{H}$ from now on and adding a parameter δ in front of $\tilde{\Delta}$ (as part of the arguments need δ to be small), in the rest of the paper, we shall study the spectral properties of the operator

$$H = \operatorname{diag}\bigl(n + j^2\bigr) + \delta\tilde{\Delta} \quad \text{on } \ell^2\bigl(\mathbb{Z}^2\bigr) \tag{1.13}$$

with $\tilde{\Delta}$ defined as in (1.12). For simplicity, we also denote $||_{\ell^1}$ by $||$.

It is known from [4] that H has pure point spectrum for all δ by using compactness arguments. However to bound the Sobolev norms, we need localization properties of the eigenfunctions of H. We have

Theorem 1. *There exist* $0 < K < \infty$, $0 < c < \infty$ *and* $0 < C < \infty$ *such that for* δ *small enough, the eigenfunctions* ϕ *of the Floquet Hamiltonian* (1.11) *with eigenvalue* E *satisfy either*

$$\left|\phi(x)\right| \leqslant C e^{-\frac{|x-(0,[E])|}{K}}, \quad \text{or} \tag{1.14}$$

$$\left|\phi(x)\right| \leqslant C \sum_{i=1}^{2} e^{-c|x-x_i|} \tag{1.15}$$

for some $x_i = (n_i, \pm j_i)$ *satisfying* $|n_i - E| \geqslant K$ *and* $|n_i + j_i^2 - E| \leqslant 2\delta$, *where* $[E]$ *is the integer part of* E.

As a direct consequence of Theorem 1, we have

Theorem 2. *Let* $s > 0$ *and* $u(0) \in H^s(\mathbb{T})$. *Then the solution* $u(t)$ *to* (1.1), (1.10) *with the initial condition* $u(0)$ *is in* $H^s(\mathbb{T})$ *for all* t *and satisfies*

$$\left\|u(t)\right\|_{H^s} \leqslant C_s \left\|u(0)\right\|_{H^s}. \tag{1.16}$$

Theorems 1 and 2 will be proved in Section 5. Theorem 2 follows from (1.14), (1.15) and a standard dyadic expansion. The main work is the proof of Theorem 1. It is an Anderson localization (A.L.) type of results in the Fourier space. The main novelty is that it holds for a *fixed* potential. In the usual A.L. setting, the potential depends on a *parameter* and localization holds on a set of parameters with large or (some times) full measure, cf. e.g., [5–7]. The reason we do not need a parameter here is because of the separation properties of the set $\{j^2, j \in \mathbb{Z}\}$. We use this to prove spacing of local eigenvalues and then *uniformly* localized eigenfunctions.

2. Pure point spectrum for the Floquet Hamiltonian

From (1.13), when $V(x,t) = 2\cos x \cos t$, the Floquet Hamiltonian H has the form

$$H = \operatorname{diag}\left(n + j^2\right) + \delta\tilde{\Delta} \quad \text{on } \ell^2\left(\mathbb{Z}^2\right), \tag{2.1}$$

where

$$\tilde{\Delta}(j, n; j', n') = \begin{cases} 1, & |j - j'| = 1 \text{ and } |n - n'| = 1, \\ 0, & \text{otherwise.} \end{cases} \tag{2.2}$$

As mentioned in Section 1, H has pure point spectrum [4]. For completeness we give a proof below to this fact using our formalism. Since δ will be taken to be small in Theorems 1 and 2, we will only address that case. (The general scheme presented in this paper extend to arbitrary δ, although some of the conditions, cf. Theorem 0 in Section 5, are verified only for small δ for the moment.)

Lemma 2.1. *H has pure point spectrum for* $|\delta| < 1/4$.

Proof. When $\delta = 0$, $\sigma(H) = \{n + j^2\} = \mathbb{Z}$ with infinite multiplicity. Let P be the projection onto the eigenspace of eigenvalue 0 and P^{c} the projection onto the complement. $P^{\mathrm{c}} = \bigoplus_{N \in \mathbb{Z}\setminus\{0\}} P_N$, where P_N is the projection onto the eigenspace of eigenvalue N, $N \neq 0$. When $\delta \neq 0$ $(0 < \delta < 1/4)$, $\sigma(H) \subseteq N + [-2\delta, 2\delta]$, $N \in \mathbb{Z}$. It is sufficient to look at spectral parameters E such that $|E| \leqslant 2\delta$.

Using the Feshbach projection or equivalently Grushin problem method, cf. e.g., [9], we have that $E \in \sigma(H)$ if and only if $0 \in \sigma(\tilde{H}_E)$, where

$$\tilde{H}_E = E - H^{00} - H^{0c}\bigl(E - H^{cc}\bigr)^{-1} H^{c0} \tag{2.3}$$

and

$$\begin{aligned}
&H^{00} = PHP = \delta\bar{\Delta},\\
&\bar{\Delta}(j, n; j'n') = \begin{cases} \tilde{\Delta}(j, n; j'n'), & (j, n), (j', n') \in \{(0, 0), (1, -1), (-1, -1)\},\\ 0, & \text{otherwise,}\end{cases}\\
&H^{0c} = PHP^{c} = \delta P\tilde{\Delta}P^{c},\\
&H^{c0} = P^{c}HP = \delta P^{c}\tilde{\Delta}P,\\
&H^{cc} = P^{c}HP^{c}.
\end{aligned} \tag{2.4}$$

We note that $\bar{\Delta}$ is of rank 3. Let $\tilde{\Delta}_{0N} = P\tilde{\Delta}P_N$ $(N \neq 0)$, then from (2.2), $\tilde{\Delta}_{0N}(j, n; j'n') \neq 0$ only if

$$N = n \pm 1 + (j \pm 1)^2|_{n+j^2=0} = \pm 2j + \begin{cases} 2,\\ 0,\end{cases} \tag{2.5}$$

and $|j - j'| = 1$ and $|n - n'| = 1$. We have

$$H^{0c} = \delta \bigoplus_{N\neq 0} \tilde{\Delta}_{0N} \tag{2.6}$$

and

$$H^{c0} = \delta \bigoplus_{N\neq 0} \tilde{\Delta}_{N0}. \tag{2.7}$$

Define

$$\begin{aligned}
A_E &= H^{00} + H^{0c}\bigl(E - H^{cc}\bigr)^{-1} H^{c0}\\
&\stackrel{\text{def}}{=} H^{00} + \delta^2 B_E \quad \bigl(|E| \leqslant 2\delta\bigr).
\end{aligned} \tag{2.8}$$

Let D be the diagonal part of H and $D_{NN} = P_N D P_N$, $N \neq 0$. Using (2.6), (2.7) and the resolvent equation, we have

$$\begin{aligned}
B_E = {}& \bigoplus_{N\neq 0} \tilde{\Delta}_{0N}(E - D_{NN})^{-1}\tilde{\Delta}_{N0}\\
&+ \Bigl[\bigoplus_{N\neq 0} \tilde{\Delta}_{0N}(E - D_{NN})^{-1}\Bigr]\delta\tilde{\Delta}^{cc}\bigl(E - H^{cc}\bigr)^{-1}\Bigl[\bigoplus_{N'\neq 0} \tilde{\Delta}_{N'0}\Bigr],
\end{aligned} \tag{2.9}$$

where $\tilde{\Delta}^{\mathrm{cc}} = P^{\mathrm{c}}\tilde{\Delta}P^{\mathrm{c}}$. B_E acts on the eigenspace $n + j^2 = 0$,

$$\tilde{\Delta}_{0N}(E - D_{NN})^{-1}(j, n; j'n') \neq 0$$

only if $n + j^2 = 0$, $|j - j'| = 1$ and $|n - n'| = 1$. Using (2.5),

$$\bigl|\tilde{\Delta}_{0N}(E - D_{NN})^{-1}(j, n; j'n')\bigr| \leqslant \mathcal{O}\bigl(1/|j|\bigr) \quad \bigl(|j| \gg 1\bigr)$$

and since

$$\bigl\|\bigl(E - H^{\mathrm{cc}}\bigr)^{-1}\bigr\| \leqslant 1 - 2\delta \quad (0 < \delta < 1/4),$$

B_E is compact for all $E \in [-2\delta, 2\delta]$. Therefore A_E is compact as H^{00} is a rank 3 operator, with 0 the only possible accumulation point. So for all $E \in [-2\delta, 2\delta]$, $\tilde{H}_E$ has pure point spectrum with E the only possible accumulation point. Clearly the above argument goes through for all $E \in N + [-2\delta, 2\delta]$, $N \in \mathbb{Z}$, with $E - N$ replacing E.

Coming back to H, $E \in \sigma(H)$ if and only if $0 \in \sigma(\tilde{H}_E)$. This implies that H has pure point spectrum with $\mathbb{Z}$ the only possible accumulation points. Here we also used the fact that the reduction in (2.3) preserves spectral multiplicity (cf. [9]). □

As explained in Section 1, in order to prove boundedness of Sobolev norms, we need to have precise localization properties of the eigenfunctions of H. For that purpose, it is essential to exhibit eigenvalue spacing. As earlier, we only need to look at $\sigma(H) \cap [-2\delta, 2\delta]$ $(0 < \delta < 1/4)$, as the eigenfunctions for the other intervals are just translates in the n direction.

When $\delta = 0$, $\sigma(H) = \{n + j^2\} = \mathbb{Z}$. From perturbation theory, the only equi-energy parabola of relevance for the spectral range $[-2\delta, 2\delta]$ is $n + j^2 = 0$ $(0 < \delta < 1/4)$. Using a Newton scheme, we compute the perturbed local eigenvalues. The result gives the necessary eigenvalue spacing in order to prove localization of eigenfunctions.

3. A Newton scheme

Let H be a linear operator on $\ell^2(\Lambda)$, $\Lambda \subseteq \mathbb{Z}^d$. We write

$$H = D + \Delta H,$$

where D is diagonal. Without loss of generality, we may assume $(\Delta H)_{ii} = 0$, for all $i \in \Lambda$. Let E be an eigenvalue of D, then $E = D_{ii}$ for some $i \in \Lambda$, with eigenfunction $u = \delta_i$. We assume

$$\begin{aligned} &D_{ii} \neq D_{jj} \quad \forall j \neq i, \quad \text{and} \\ &\|\Delta H\| < \frac{1}{2} \inf_{j \neq i} |D_{ii} - D_{jj}|. \end{aligned} \tag{3.1}$$

We call i the *resonant* site. Let $\mathcal{R} = \{i\}$. $\mathcal{R}^{\mathrm{c}} = \Lambda \setminus \{i\}$. We compute the eigenvalues and eigenfunctions using the following iteration scheme.

Remark. Under the assumption (3.1), i is the only resonant site, cf. (3.9). The scheme below however can be applied to cases where there is symmetry, (3.1) is violated and there are more than 1 resonant site. For example it can be used to compute eigenvalue splitting for the (time independent) periodic Schrödinger operator in 1d. We leave this aspect of things to a future publication. The approach here is different from the Raleigh–Schrödinger scheme in quantum mechanics. It is closer to the Grushin–Feshbach effective operator method. More precisely it provides a way to compute eigenvalues when the effective operator is finite-dimensional.

We seek solutions to the eigenvalue problem

$$(H-E)u=0, \tag{3.2}$$

such that $u|_{\mathcal{R}}=1$ is fixed. As a zeroth order approximation,

$$E=D_{ii}, \qquad u=\delta_i. \tag{3.3}$$

So

$$(H-E)u=\Delta H\delta_i \stackrel{\text{def}}{=} F(u), \tag{3.4}$$

where $F(u)$ is the error satisfying

$$F(u)|_{\mathcal{R}}=0, \qquad F(u)=F(u)|_{\mathcal{R}^c}.$$

Assume we have (3.2), (3.3) at the nth iteration, with $u=u^{(n)}$, $E=E^{(n)}$. To obtain the $(n+1)$th approximant, we write

$$\begin{aligned} u^{(n+1)} &= u^{(n)}+\Delta u^{(n+1)} \stackrel{\text{def}}{=} u+\Delta u, \\ E^{(n+1)} &= E^{(n)}+\Delta E^{(n+1)} \stackrel{\text{def}}{=} E+\Delta E, \end{aligned} \tag{3.5}$$

such that

$$\bigl(H-E^{(n)}\bigr)u^{(n+1)}\bigr|_{\mathcal{R}^c}=(H-E)(u+\Delta u)|_{\mathcal{R}^c}=0, \tag{3.6}$$

$$\bigl(H-E^{(n+1)}\bigr)u^{(n+1)}\bigr|_{\mathcal{R}}=(H-E-\Delta E)(u+\Delta u)|_{\mathcal{R}}=0 \tag{3.7}$$

are verified.

Since $\Delta u|_{\mathcal{R}}=0$, from (3.6),

$$(H-E)|_{\mathcal{R}^c}\Delta u|_{\mathcal{R}^c}=-(H-E)u|_{\mathcal{R}^c}=-F(u)|_{\mathcal{R}^c}, \tag{3.8}$$

so

$$\Delta u|_{\mathcal{R}^c}=-(H|_{\mathcal{R}^c}-E)^{-1}F(u)|_{\mathcal{R}^c}. \tag{3.9}$$

From (3.7),

$$(H-E)u|_{\mathcal{R}}+(H-E)\Delta u|_{\mathcal{R}}-\Delta Eu|_{\mathcal{R}}-\Delta E\Delta u|_{\mathcal{R}}=0.$$

The first term is 0 from (3.7), the fourth term is 0 since $\Delta u|_{\mathcal{R}} = 0$. So

$$\Delta E = (H - E)\Delta u|_{\mathcal{R}} = (\Delta H \Delta u)|_{\mathcal{R}}, \tag{3.10}$$

where we used $u|_{\mathcal{R}} = 1$ by definition. The error of approximation

$$\begin{aligned} F(u + \Delta u) &= (H - E - \Delta E)(u + \Delta u) \\ &= (H - E - \Delta E)(u + \Delta u)|_{\mathcal{R}^c} \quad \text{from (3.7)} \\ &= -\Delta E(u + \Delta u)|_{\mathcal{R}^c}. \end{aligned} \tag{3.11}$$

We now show that the above iteration scheme converges for H in (2.1) restricted to appropriate subsets of $\mathbb{Z}^2$.

Convergence of the Newton scheme. It is sufficient to look at $\sigma(H) \cap [-2\delta, 2\delta]$ $(0 < \delta < 1/4)$. $n + j^2 = 0$ is the resonant parabola. Let $P = \{(j, n) \mid n + j^2 = 0\}$. $\mathbb{Z}^2 \setminus P$ are non-resonant. For any two points $(j, n), (j', n') \in P$, $(j, n) \neq (j', n')$, $|(j, n) - (j', n')|_\infty \geqslant d$ with

$$\begin{aligned} d &= \max\bigl(\bigl||j'| - |j|\bigr| \bigl(|j'| + |j|\bigr), |j - j'|\bigr) \\ &= \begin{cases} 2|j|, & |j| = |j'|,\ j \neq j', \\ (|j'| - |j|)(|j'| + |j|), & |j| \neq |j'|. \end{cases} \end{aligned} \tag{3.12}$$

For all $(j, n) \in P$, $|j| > 1$, define

$$\Lambda_j = \bigl\{(j', n') \bigm| \bigl|(j', n') - (j, n)\bigr|_\infty \leqslant L_j\bigr\}, \quad |j| \leqslant L_j \leqslant 2\bigl(|j| - 1\bigr) \tag{3.13}$$

to be the square centered at (j, n) with side-length $2L_j$. From (3.12), $P \cap \Lambda_j = \{(j, n)\}$, $|j| > 1$. So (j, n) is the only resonant site in Λ_j at $E = 0$.

For any $S \subset \mathbb{Z}^2$, define

$$\begin{aligned} &H_S(j', n'; j'', n'') = H(j', n'; j'', n''), \quad (j', n'), (j'', n'') \in S, \\ &H_S = 0, \quad \text{otherwise}. \end{aligned} \tag{3.14}$$

We now prove that the Newton scheme in (3.2)–(3.11) converges for H_{Λ_j}, when $|j|$ is sufficiently large.

For simplicity of notation, we write H for H_{Λ_j}. $\mathcal{R} = \{(j, n)\}$, $(j, n) \in P$, $\mathcal{R}^c = \Lambda_j \setminus \{(j, n)\}$. $H_{\mathcal{R}^c}$ is H restricted to $\mathcal{R}^c$. Define

$$F^{(k)}\bigl(u^{(k)}\bigr) \stackrel{\text{def}}{=} \bigl(H - E^{(k)}\bigr)u^{(k)}. \tag{3.15}$$

From (3.11)

$$F^{(k)}\bigl(u^{(k)}\bigr) = -\Delta E^{(k)} u^{(k)}\bigr|_{\mathcal{R}^c}. \tag{3.16}$$

From (3.9), (3.10)

$$\Delta u^{(k+1)}\big|_{\mathcal{R}^c} = -\big(H_{\mathcal{R}^c} - E^{(k)}\big)^{-1} F^{(k)}\big(u^{(k)}\big)\big|_{\mathcal{R}^c}, \tag{3.17}$$

$$\Delta E^{(k)} = \delta \sum_{\substack{|j-j'|=1\\|n-n'|=1}} \Delta u^{(k)}(j',n'), \tag{3.18}$$

where we have put back the superscript according to (3.5).

The following lemma shows that the Newton scheme (3.15)–(3.18) converges exponentially fast for $|j|$ sufficiently large.

Lemma 3.1.

$$\big\|F^{(k)}\big\| \leqslant \frac{2C^2\|F^{(0)}\|}{|j|}\big\|F^{(k-1)}\big\| < \frac{1}{2}\big\|F^{(k-1)}\big\| \quad \textit{for all } k \geqslant 1, \tag{3.19}$$

$$\big|\Delta E^{(k)}\big| < \frac{C}{|j|}\big\|F^{(k-1)}\big\| \quad \textit{for all } k \geqslant 1, \tag{3.20}$$

where $F^{(k)} = F^{(k)}(u^{(k)})$, $1 < C \leqslant 1/(1-2\delta)$ $(0 < \delta < 1/4)$, $u^{(0)} = \delta_{(j,n)}$, $E^{(0)} = 0$, *provided* $|j| > 4C^2\|F^{(0)}\| \stackrel{\text{def}}{=} j_0$.

Remark. (3.17)–(3.20) show that the above iteration scheme provides a convergent series expansion for the eigenvalue of H_{Λ_j} in $[-2\delta, 2\delta]$ and its eigenfunction, although for the purpose of this paper, this is not needed.

Proof. We start from $k = 1$. From (3.17),

$$\Delta u^{(1)}\big|_{\mathcal{R}^c} = -\big(H_{\mathcal{R}^c} - E^{(0)}\big)^{-1} F^{(0)}\big|_{\mathcal{R}^c}, \tag{3.21}$$

so

$$\big\|\Delta u^{(1)}\big\| \leqslant \big\|F^{(0)}\big\|, \tag{3.22}$$

where we used

$$\big\|H_{\mathcal{R}^c} - E^{(0)}\big\| \geqslant 1. \tag{3.23}$$

From (3.18),

$$\begin{aligned}
\Delta E^{(1)} &= \delta \sum_{\substack{|j-j'|=1\\|n-n'|=1}} \Delta u^{(1)}(j',n') \\
&= \delta \sum_{\substack{|j-j'|=1\\|n-n'|=1}} \big[\big(D_{\mathcal{R}^c} - E^{(0)}\big)^{-1} F^{(0)}\big](j',n') \\
&\quad + \big[\big(D_{\mathcal{R}^c} - E^{(0)}\big)^{-1} \delta\tilde{\Delta}\big(H_{\mathcal{R}^c} - E^{(0)}\big)^{-1} F^{(0)}\big](j',n'),
\end{aligned} \tag{3.24}$$

so

$$\left|\Delta E^{(1)}\right| \leqslant \frac{C}{|j|}\left\|F^{(0)}\right\|, \tag{3.25}$$

where we used

$$\left|n' + j'^2\right| = \left|n \pm 1 + (j \pm 1)^2\right| \geqslant |j| \quad \big(|j| > 1\big) \tag{3.26}$$

and (3.23).

Using (3.25), (3.22) in (3.16), we have

$$\left\|F^{(1)}\right\| \leqslant \frac{C}{|j|}\left\|F^{(0)}\right\| \cdot \left\|F^{(0)}\right\|. \tag{3.27}$$

So

$$\left\|F^{(1)}\right\| < \frac{1}{2}\left\|F^{(0)}\right\|, \tag{3.28}$$

if $|j| > 2C\|F^{(0)}\|$. (3.25), (3.28) show that (3.19), (3.20) hold at $k = 1$. (3.25) shows that

$$\left\|H_{\mathcal{R}^c} - E^{(1)}\right\| > 1/C. \tag{3.29}$$

Assume (3.19), (3.20) hold for all $k \leqslant K$, which implies also that

$$\left\|H_{\mathcal{R}^c} - E^{(K)}\right\| > 1/C. \tag{3.30}$$

From (3.17)

$$\Delta u^{(K+1)}\big|_{\mathcal{R}^c} = -\big(H_{\mathcal{R}^c} - E^{(K)}\big)^{-1} F^{(K)}\big|_{\mathcal{R}^c}, \tag{3.31}$$

so

$$\left\|\Delta u^{(K+1)}\big|_{\mathcal{R}^c}\right\| \leqslant C\left\|F^{(K)}\right\| \tag{3.32}$$

using (3.30). From (3.18)

$$\begin{aligned}
\Delta E^{(K+1)} &= \delta \sum_{\substack{|j-j'|=1\\|n-n'|=1}} \Delta u^{(K+1)}(j', n') \\
&= -\delta \sum_{\substack{|j-j'|=1\\|n-n'|=1}} \big[\big(D_{\mathcal{R}^c} - E^{(K)}\big)^{-1} F^{(K)}\big](j', n') \\
&\quad + \big[\big(D_{\mathcal{R}^c} - E^{(K)}\big)^{-1} \delta\tilde{\Delta}\big(H_{\mathcal{R}^c} - E^{(K)}\big)^{-1} F^{(K)}\big](j', n').
\end{aligned}$$

So

$$\left|\Delta E^{(K+1)}\right| \leqslant \frac{C}{|j|}\left\|F^{(K)}\right\|. \tag{3.33}$$

Hence

$$\left\|H_{\mathcal{R}^c} - E^{(K+1)}\right\| > 1/C. \tag{3.34}$$

Using (3.33), (3.32), (3.17) in (3.16), we have

$$\left\|F^{(K+1)}\right\| \leqslant \frac{C}{|j|}\left\|F^{(K)}\right\| \cdot C\left(\sum_{k=0}^{K}\left\|F^{(k)}\right\|\right), \tag{3.35}$$

$$\left\|F^{(K+1)}\right\| \leqslant \frac{2C^2}{|j|}\left\|F^{(K)}\right\| \cdot \left\|F^{(0)}\right\|, \tag{3.36}$$

if $|j| > 4C^2\|F^{(0)}\| \stackrel{\text{def}}{=} j_0$, where we used (3.19) from $1 \leqslant k \leqslant K$ to estimate the sum in (3.35). (3.33), (3.34), (3.36) imply that the lemma holds by induction. $\square$

4. Computation of local eigenvalues and eigenfunctions

Let $P = \{(j,n) \mid n + j^2 = 0\}$ be the resonant parabola at $E = 0$. Let Λ_j be the square defined as in (3.13). We use the convergent Newton scheme to compute the eigenvalue $E \in \sigma(H_{\Lambda_j}) \cap [-2\delta, 2\delta]$ $(0 < \delta < 1/4)$.

Let $u^{(0)} = \delta_{(j,n)}$, $E^{(0)} = 0$. Then

$$F^{(0)}(j',n') = F^{(0)}\big(u^{(0)}\big)(j',n') = \begin{cases} \delta, & |j - j'| = 1, \text{ and } |n - n'| = 1, \\ 0, & \text{otherwise.} \end{cases} \tag{4.1}$$

Assume $|j| > 4C^2\|F^{(0)}\| = j_0$, so that Lemma 3.1 is applicable.

Lemma 4.1. *E has the convergent series expansion*

$$E = -\frac{\delta^2}{j^2} + \frac{a_4\delta^4}{j^4} + \cdots, \tag{4.2}$$

where a_4 is independent of the L_j in (3.13).

Proof. From (3.17),

$$\Delta u^{(1)}\big|_{\mathcal{R}^c} = -\big(H_{\mathcal{R}^c} - E^{(0)}\big)^{-1} F^{(0)}\big|_{\mathcal{R}^c}. \tag{4.3}$$

In view of (3.18), we compute $\Delta u^{(1)}$ on the set

$$\big\{(j',n') \bigm| |n - n'| = 1 \text{ and } |j - j'| = 1\big\}. \tag{4.4}$$

We have

$$\begin{aligned}\Delta u^{(1)}(j',n') = &-\big(D-E^{(0)}\big)^{-1}F^{(0)}(j',n')\\ &-\big(D-E^{(0)}\big)^{-1}\delta\tilde{\Delta}\big(D-E^{(0)}\big)^{-1}F^{(0)}(j',n')\\ &-\big(D-E^{(0)}\big)^{-1}\delta\tilde{\Delta}\big(D-E^{(0)}\big)^{-1}\delta\tilde{\Delta}\big(D-E^{(0)}\big)^{-1}F^{(0)}(j',n')\\ &-\big(D-E^{(0)}\big)^{-1}\delta\tilde{\Delta}\big(D-E^{(0)}\big)^{-1}\delta\tilde{\Delta}\big(D-E^{(0)}\big)^{-1}\delta\tilde{\Delta}\big(D-E^{(0)}\big)^{-1}F^{(0)}(j',n')\\ &-\big(D-E^{(0)}\big)^{-1}\delta\tilde{\Delta}\big(D-E^{(0)}\big)^{-1}\delta\tilde{\Delta}\big(D-E^{(0)}\big)^{-1}\delta\tilde{\Delta}\\ &\quad\big(D-E^{(0)}\big)^{-1}\delta\tilde{\Delta}\big(D-E^{(0)}\big)^{-1}F^{(0)}(j',n')\\ &-\big(D-E^{(0)}\big)^{-1}\delta\tilde{\Delta}\big(D-E^{(0)}\big)^{-1}\delta\tilde{\Delta}\big(D-E^{(0)}\big)^{-1}\delta\tilde{\Delta}\big(D-E^{(0)}\big)^{-1}\delta\tilde{\Delta}\\ &\quad\big(D-E^{(0)}\big)^{-1}\delta\tilde{\Delta}\big(H_{\mathcal{R}^c}-E^{(0)}\big)^{-1}F^{(0)}(j',n').\end{aligned}\tag{4.5}$$

In view of the support of $F^{(0)}$: supp $F^{(0)}$ and $\tilde{\Delta}$, the second and fourth terms in (4.5) are zero on the set in (4.4), the last two terms are of order $\mathcal{O}(1/|j|^3)$. So

$$\begin{aligned}\Delta u^{(1)}(j',n') = &-\big(D-E^{(0)}\big)^{-1}F^{(0)}(j',n')\\ &-\delta^2\big(D-E^{(0)}\big)^{-1}\tilde{\Delta}\big(D-E^{(0)}\big)^{-1}\tilde{\Delta}\big(D-E^{(0)}\big)^{-1}F^{(0)}(j',n')\\ &+\mathcal{O}\big(1/|j|^3\big).\end{aligned}\tag{4.6}$$

The second term is in fact of order $\mathcal{O}(1/|j|^3)$. Assume $n'=n-1$, $j'=j-1$. The second term gives

$$\begin{aligned}&\big(D-E^{(0)}\big)^{-1}\tilde{\Delta}\big(D-E^{(0)}\big)^{-1}\tilde{\Delta}\big(D-E^{(0)}\big)^{-1}F^{(0)}(j',n')\\ &\quad=\frac{1}{n-1+(j-1)^2}\frac{1}{n-2+j^2}\bigg(\frac{1}{n-1+(j-1)^2}+\frac{1}{n-1+(j+1)^2}\bigg)+\mathcal{O}\big(1/|j|^3\big)\\ &\quad=\mathcal{O}\big(1/|j|^3\big)\end{aligned}\tag{4.7}$$

as claimed. Similar estimates holds for the other (j',n') in (4.4). (3.18) gives

$$\begin{aligned}\Delta E^{(1)} &= \delta\sum_{\substack{|n'-n|=1\\|j'-j|=1}}\big(D-E^{(0)}\big)^{-1}F^{(0)}(j',n')+\mathcal{O}\big(1/|j|^3\big)\\ &=\delta^2\bigg[\frac{1}{n+1+(j-1)^2}+\frac{1}{n+1+(j+1)^2}+\frac{1}{n-1+(j+1)^2}+\frac{1}{n-1+(j-1)^2}\bigg]\\ &\quad+\mathcal{O}\big(1/|j|^3\big)\\ &=-\frac{\delta^2}{j^2}+\mathcal{O}\bigg(\frac{1}{|j|^3}\bigg).\end{aligned}\tag{4.8}$$

The $\mathcal{O}(1/|j|^3)$ is in fact absent from symmetry arguments. Using (3.16), (3.17), (4.8) is the only contribution at order $\mathcal{O}(1/j^2)$. So E has the expansion in (4.2). □

5. Proof of the theorems

Let E be an eigenvalue of H with eigenfunction ϕ:

$$(H-E)\phi = 0, \tag{5.1}$$

$\phi \in \ell^2(\mathbb{Z}^2)$ from Lemma 2.1. Write $x = (j, n) \in \mathbb{Z}^2$. We prove

Theorem 0. *Assume*

(H1) $\exists L_0 \gg 1$, *such that* $\operatorname{dist}(0, \sigma(H_{\Lambda_0}) \setminus \{0\}) \gg e^{-L_0}$, *where* $\Lambda_0 = [-L_0^2+1, L_0^2-1]^2$,
(H2) $\exists \ell_0$, $0 < \ell_0 \leqslant L_0$ *such that for* $|j| > \ell_0$, *Lemma* 3.1 *is available,*
(H3) *if* $0 \in \sigma(H_{\Lambda_0})$, *then* $0 \in \sigma(H_\Lambda)$ *for all* $\Lambda = [-L, L]^2 \supset \Lambda_0$ *with the same multiplicity.*

Under the conditions (H1)–(H3), *there exist* $0 < K < \infty$, $0 < C < \infty$, $0 < c < \infty$ (*depending only on* δ, L_0), *such that*

$$\text{either} \quad \left|\phi(x)\right| \leqslant C e^{-\frac{|x-(0,E)|}{K}}, \tag{5.2}$$

$$\text{or} \quad \left|\phi(x)\right| \leqslant C \sum_{i=1}^{2} e^{-c|x-x_i|} \tag{5.3}$$

for some $x_i = (n_i, \pm j_i)$ *satisfying*

$$|n_i - E| \geqslant K \quad \text{and} \quad \left|n_i + j_i^2 - E\right| \leqslant 2\delta. \tag{5.4}$$

Remark. If (H1) holds for the cube $\Lambda_0 = [-L_0^2+1, L_0^2-1]^2$, then it holds for the cubes $\Lambda' = [-L_0^2+L, L_0^2-L]^2$, $(0 \leqslant L \leqslant L_0)$.

Proof of the theorem. As earlier, we may assume $E \in [-2\delta, 2\delta]$ $(0 < \delta < 1/4)$ without loss of generality. This is because if we define $\bar\phi$ to be $\bar\phi(\cdot,\cdot) = \phi(\cdot - N, \cdot)$, $N \in \mathbb{Z}$. Then $(H - \bar E)\bar\phi = 0$ with $\bar E = N + E$. So $\bar\phi$ has the same localization properties as ϕ.

Let

$$\Lambda = \left[-L^2+1, L^2-1\right]^2, \quad L > L_0. \tag{5.5}$$

So $\Lambda \supset \Lambda_0$. Let $P = \{(j,n) \mid n + j^2 = 0\}$ be the set of resonant points. Let $P_\Lambda = P \cap \Lambda$. We cover P_Λ with Λ_0, $\Lambda_{\pm j}$ $(L_0 \leqslant j < L)$, $\Lambda_{\pm j}$ as defined in (3.13). Denote by μ the eigenvalues of H_{Λ_0} in $[-2\delta, 2\delta]$, and $\lambda_{\pm j}$ the eigenvalues of $H_{\Lambda_{\pm j}}$ in $[-2\delta, 2\delta]$, i.e., $\{\mu\} = \sigma(H_{\Lambda_0}) \cap [-2\delta, 2\delta]$, $\lambda_{\pm j} = \sigma(H_{\Lambda_{\pm j}}) \cap [-2\delta, 2\delta]$ $(L_0 \leqslant j < L)$.

Below we show that uniformly in Λ, μ, $\lambda_{\pm j}$ are approximate eigenvalues of H_Λ in $[-2\delta, 2\delta]$. (For precise meaning of this, see (5.16), (5.17).) Therefore as $\Lambda \nearrow \mathbb{Z}^2$, the nonzero but "small" eigenvalues of H_Λ "come" from $H_{\Lambda_{\pm j}}$ $(L_0 \leqslant j < L)$ (see (5.18), (5.19)).

Let Λ_* denotes either Λ_0 or $\Lambda_{\pm j}$ and

$$\partial\Lambda_* = \big\{(j',n')\in\Lambda_* \,\big|\, \exists (j'',n'')\in\Lambda\setminus\Lambda_*,\ \text{such that } |j'-j''|=1 \text{ and } |n'-n''|=1\big\} \tag{5.6}$$

be the interior boundary of Λ_* relative to Λ. Let $P_{\Lambda_*} = P\cap\Lambda_*$.

Define $\Lambda_*' = \Lambda_*\setminus P_{\Lambda_*}$. We have

$$\operatorname{dist}\big(\sigma(H_{\Lambda_*'}), [-2\delta, 2\delta]\big) \geqslant 1-2\delta \quad (0<\delta<1/4). \tag{5.7}$$

Let $H_{\Lambda_*'}$ be defined as in (3.14) and

$$\Gamma = H_{\Lambda_*} - H_{\Lambda_*'}. \tag{5.8}$$

Assume ϕ is the eigenfunction for the eigenvalue $\lambda_* \in \sigma(H_{\Lambda_*})\cap[-2\delta, 2\delta]$:

$$(H_{\Lambda_*} - \lambda_*)\phi = 0. \tag{5.9}$$

We have the identity

$$\phi = -(H_{\Lambda_*'} - \lambda_*)^{-1}\Gamma\phi \tag{5.10}$$

using (5.8). Using (5.6), (5.7), we then have for some $\alpha>0$ (depending only on δ)

$$\begin{aligned} &\|\phi\|_{\ell^2(\partial\Lambda_0)} \leqslant e^{-\alpha L_0} \quad (L_0 \text{ large enough}), \quad \text{and}\\ &\|\phi\|_{\ell^2(\partial\Lambda_j)} \leqslant e^{-\alpha|j|}. \end{aligned} \tag{5.11}$$

Let H_Λ be defined as in (3.14) and

$$\tilde\Gamma = H_\Lambda - H_{\Lambda_*}. \tag{5.12}$$

We compute

$$(H_\Lambda - \lambda_*)\phi = (H_{\Lambda_*} - \lambda_*)\phi + \tilde\Gamma\phi. \tag{5.13}$$

So

$$\big\|(H_\Lambda - \lambda_*)\phi\big\| \leqslant \|\tilde\Gamma\phi\| \leqslant \begin{cases} e^{-\alpha L_0} & \text{if } \Lambda_* = \Lambda_0,\\ e^{-\alpha|j|} & \text{if } \Lambda_* = \Lambda_{\pm j}, \end{cases} \tag{5.14}$$

where we used (5.11). This shows that $\exists\lambda\in\sigma(H_\Lambda)\cap[-2\delta, 2\delta]$ such that

$$|\lambda_* - \lambda| \leqslant \begin{cases} e^{-\alpha L_0} & \text{if } \lambda_* \in \sigma(H_{\Lambda_0})\cap[-2\delta, 2\delta],\\ e^{-\alpha|j|} & \text{if } \lambda_* \in \sigma(H_{\pm j})\cap[-2\delta, 2\delta]. \end{cases} \tag{5.15}$$

Let $I = |P_\Lambda|$ be the number of resonant points in Λ. We further label the (normalized) local eigenfunctions in (5.9) as $\phi_1, \phi_2, \dots, \phi_I$. Using (5.7), (5.10), the matrix A with entries $A_{ij} = (\phi_i, \phi_j)$ is invertible. Combined with (5.15), this shows that

$$\lambda = \mu + \mathcal{O}\big(e^{-\alpha L_0}\big) \quad \text{or} \tag{5.16}$$

$$\lambda = \lambda_{\pm j} + \mathcal{O}\big(e^{-\alpha |j|}\big), \tag{5.17}$$

where μ are the eigenvalues of H_{Λ_0} in $[-2\delta, 2\delta]$ and $\lambda_{\pm j}$ are the eigenvalues of $H_{\Lambda_{\pm j}}$ in $[-2\delta, 2\delta]$ and have the convergent series expansion as in (4.2).

Let $\mu_{\min}$ be the smallest (in absolute value) nonzero eigenvalue of H_{Λ_0} in $[-2\delta, 2\delta]$,

$$K' = \delta/\sqrt{|\mu_{\min}|} > 0 \tag{5.18}$$

and set

$$\Lambda = \big[-K'^2 + K', K'^2 + K'\big]^2. \tag{5.19}$$

(H1), (H3), (4.2), (5.16), (5.17) then imply

$$\operatorname{dist}\big[\sigma(H_\Lambda) \cap [-2\delta, 2\delta], \sigma(H_{\Lambda_{\pm j}}) \cap [-2\delta, 2\delta]\big] \geqslant \frac{\mathcal{O}(\delta^2)}{|j|^3} \quad \text{for } j > K'. \tag{5.20}$$

Given $E \in [-2\delta, 2\delta]$, we define:

- E resonant with $\Lambda_{\pm j}$, if

$$|E - \lambda_{\pm j}| \leqslant e^{-j^\rho} \quad (j > K', 0 < \rho < 1/2), \tag{5.21}$$

 where $\Lambda_{\pm j}$ are as defined in (3.13), $\lambda_{\pm j}$ are the eigenvalues of $H_{\Lambda_{\pm j}}$ in $[-2\delta, 2\delta]$ and have the convergent series expansion in (4.2).
- E resonant with Λ (defined in (5.19)) if

$$\operatorname{dist}\big(E, \sigma(H_\Lambda)\big) \leqslant e^{-K'^\rho} \tag{5.22}$$

 and

$$|E - \lambda_{\pm j}| > e^{-j^\rho} \quad (0 < \rho < 1/2),\ \forall j > K'. \tag{5.23}$$

Otherwise we say that E is non-resonant. Since for all $E \in [-2\delta, 2\delta]$, either (5.21) holds for some $\Lambda_{\pm j}$ or (5.23) holds, if (5.23) is satisfied and (5.22) is not, then from (5.19), E is not an eigenvalue of $\sigma(H)$ as

$$\operatorname{dist}\big(\sigma(H), \sigma(H_\Lambda)\big) \leqslant e^{-\mathcal{O}(K')} \ll e^{-K'^\rho} \quad (0 < \rho < 1/2).$$

So if E is non resonant and $E \in \sigma(H) \cap [-2\delta, 2\delta]$, then E is an accumulation point since $\sigma(H)$ is pure point from Lemma 2.1. In this case $E = 0$.

If E is resonant with Λ_j $(j > K')$, then

$$|E| \geqslant \frac{\mathcal{O}(\delta^2)}{j^2} \tag{5.24}$$

and

$$\operatorname{dist}\bigl(E,\sigma(H_\Lambda)\setminus\{0\}\bigr)\geqslant\frac{\mathcal{O}(\delta^2)}{K'^3}\quad (j>K') \tag{5.25}$$

from (5.18), (5.19), (5.21), and for $|j'|\neq|j|$, $|j'|>K'$

$$\begin{aligned}\operatorname{dist}\bigl(E,\sigma(H_{\Lambda_{\pm j'}})\bigr)&\geqslant\left|\frac{\mathcal{O}(\delta^2)}{j^2}-\frac{\mathcal{O}(\delta^2)}{j'^2}\right|-\mathcal{O}\bigl(e^{-j^\rho}\bigr)-\mathcal{O}\bigl(e^{-j'^\rho}\bigr)\\&\geqslant\frac{\mathcal{O}(\delta^2)}{[\min(|j|,|j'|)]^3},\end{aligned} \tag{5.26}$$

where the $\mathcal{O}(\delta^2)$ are uniform in $|j|,|j'|>K'$. (5.24)–(5.26) together with (H1), (H3) imply that

$$\operatorname{dist}\bigl(E,\sigma(H_{\Lambda'\setminus\{(\pm j,n)\}})\bigr)\geqslant\frac{\mathcal{O}(\delta^2)}{|j|^3}\quad\bigl(|j|>K'\bigr) \tag{5.27}$$

for all $\Lambda'\subseteq\mathbb{Z}^2$, where the $\mathcal{O}(\delta^2)$ is uniform in Λ'.

If E is resonant with Λ, then since

$$|E-\lambda_{\pm j}|>e^{-j^\rho}\quad (0<\rho<1/2),\ \forall j>K', \tag{5.28}$$

by definition,

$$\operatorname{dist}\bigl(E,\sigma(H_{\Lambda'\setminus\Lambda})\bigr)\geqslant\mathcal{O}\bigl(e^{-J^\rho}\bigr),\quad \forall\Lambda'=-\bigl[J^2,J^2\bigr]^2,\ J>K'. \tag{5.29}$$

Assume E is resonant with $\Lambda_{\pm j}$. Let

$$\Lambda'(k)=(j,n)+[-k,k]^2,\quad j>0,\ n+j^2=0,\ k\geqslant 2(j-1), \tag{5.30}$$

be cubes centered at (j,n) with side-lengths $2k$. Let $P=\{(j',n')\mid n'+j'^2=0\}$ as before. Assume $\Lambda'(k)$ is such that

$$\operatorname{dist}\bigl[(j',n'),\partial\Lambda'(k)\bigr]\geqslant|j'| \tag{5.31}$$

for all $(j',n')\in P\cap\Lambda'(k)$, where $\partial\Lambda'(k)$ is the boundary of $\Lambda'(k)$ defined as in (5.6). Assume also that if $\Lambda'(k)\cap\Lambda\neq\emptyset$, where Λ is defined in (5.18), (5.19), then $\Lambda'(k)\supset\Lambda$. Similarly we define

$$\Lambda''(k)=(-j,n)+[-k,k]^2,\quad j^2+n=0. \tag{5.32}$$

Let $\mathcal{R}=\{(j,n),(-j,n)\}$, $j^2+n=0$. Given any $x\in\mathbb{Z}^2$, it is easy to see that there exists k, such that either $x\in\Lambda'(k)$ or $x\in\Lambda''(k)$ or both, and

$$10\,\mathrm{dist}\big(x, \mathcal{R}\cap\Lambda'(k)\big) \geqslant \mathrm{dist}\big(x, \partial\Lambda'(k)\big) \geqslant \mathrm{dist}\big(x, \mathcal{R}\cap\Lambda'(k)\big), \quad \text{if } x\in\Lambda'(k), \tag{5.33}$$

$$10\,\mathrm{dist}\big(x, \mathcal{R}\cap\Lambda''(k)\big) \geqslant \mathrm{dist}\big(x, \partial\Lambda''(k)\big) \geqslant \mathrm{dist}\big(x, \mathcal{R}\cap\Lambda''(k)\big), \quad \text{if } x\in\Lambda''(k). \tag{5.34}$$

Assume $x\in\Lambda'(k)$ ($x\in\Lambda''(k)$ works in the same way). We write Λ' for $\Lambda'(k)$ for simplicity. Let $H_{\Lambda'\backslash\mathcal{R}}$ be defined as in (3.14) and

$$\Gamma = H - H_{\Lambda'\backslash\mathcal{R}}. \tag{5.35}$$

Then

$$\phi(x) = \sum G_{\Lambda'\backslash\mathcal{R}}(x,y)[\Gamma\phi](y), \tag{5.36}$$

where ϕ is the eigenfunction of H with eigenvalue E ($\phi\neq 0$) and

$$G_{\Lambda'\backslash\mathcal{R}}(x,y) = (H_{\Lambda'\backslash\mathcal{R}} - E)^{-1}(x,y). \tag{5.37}$$

To estimate $G_{\Lambda'\backslash\mathcal{R}}$, we use the resolvent equation. There are two cases, $\Lambda\cap\Lambda' = \emptyset$ and $\Lambda\subset\Lambda'$. When $\Lambda\cap\Lambda'=\emptyset$, we cover $\Lambda'\setminus\mathcal{R}$ with cubes $\bar\Lambda$ and annulus $A=\bar\Lambda\setminus\{(\pm j,n)\}$, such that either (a) $\bar\Lambda\cap P=\emptyset$ or (b) $\bar\Lambda\cap P=(j',n')$ and

$$\mathrm{dist}\big[(j',n'),\partial\bar\Lambda\big] \geqslant |j'|/4. \tag{5.38}$$

In case (a),

$$\mathrm{dist}\big(E,\sigma(H_{\bar\Lambda})\big) = \mathcal{O}(1), \tag{5.39}$$

in case (b),

$$\mathrm{dist}\big(E,\sigma(H_{\bar\Lambda})\big) \geqslant \frac{\mathcal{O}(\delta^2)}{[\min(|j|,|j'|)]^3}, \tag{5.40}$$

similar to (5.26). In case (a), we have

$$\big|(H_{\bar\Lambda}-E)^{-1}(x,y)\big| \leqslant Ce^{-\kappa|x-y|}, \quad C,\kappa>0, \tag{5.41}$$

for all x and y. In case (b), we have

$$\big|(H_{\bar\Lambda}-E)^{-1}(x,y)\big| \leqslant Ce^{-\kappa|x-y|}, \tag{5.42}$$

for all x and y such that $|x-y|\geqslant\sqrt{j'}$, j' large by Neumann series.

Iterating the resolvent equation using (5.41), (5.42), (5.27), we obtain

$$\big|G_{\Lambda'\backslash\mathcal{R}}(x,y)\big| \leqslant e^{-\kappa'|x-y|}, \quad 0<\kappa'<\kappa, \tag{5.43}$$

for all Λ', x, $y\in\Lambda'$, $|x-y|\geqslant k/10$. When Λ' is such that $\Lambda'\cap P=\{(j,n)\}$, then

$$\left\|(H_{\Lambda'\setminus\mathcal{R}}-E)^{-1}\right\|=\mathcal{O}(1), \tag{5.44}$$

$$\left|G_{\Lambda'\setminus\mathcal{R}}(x,y)\right|\leqslant e^{-\kappa'|x-y|} \tag{5.45}$$

for all x and y.

When $\Lambda\subset\Lambda'$, we cover $\Lambda'\setminus\mathcal{R}$ with Λ, annulus A and cubes $\bar{\Lambda}$ satisfying properties (a), (b) as before. Using (5.41), (5.42), (5.27) in the resolvent equation, we obtain (5.43), assuming $|j|>K>K'$. (5.36), (5.43), (5.45) then give (5.3).

Using (5.28), (5.29) in place of (5.26), (5.27) in the resolvent iteration, we obtain (5.2) in the same way using (5.36) with Λ replacing $\mathcal{R}$. $\square$

Proof of Theorem 1. We only need to verify that (H1), (H3) are satisfied as Lemma 3.1 is available. This is where we need δ to be small. Take $\Lambda=[-8,8]^2$. From (2.3), $E\in\sigma(H_\Lambda)$ if and only if $0\in\sigma(\tilde{H}_{\Lambda,E})$, where $\tilde{H}_{\Lambda,E}$ is defined similarly as in (2.3), (2.4) with H_Λ replacing H in (2.4). $\tilde{H}_{\Lambda,E}$ is a 5×5 matrix. Specifically we have

$$\begin{aligned}\tilde{H}_{\Lambda,E}&=E-\delta\begin{pmatrix}0&1&1&0&0\\1&0&0&0&0\\1&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix}\\&\quad+\delta^2\begin{pmatrix}1&0&0&0&0\\0&1/4&-1/2&0&0\\0&-1/2&1/4&0&0\\0&0&0&1/4&0\\0&0&0&0&1/4\end{pmatrix}+\mathcal{O}(\delta^3)\\&\stackrel{\text{def}}{=}E-A+\mathcal{O}(\delta^3).\end{aligned} \tag{5.46}$$

It is easy to verify that the 5 eigenvalues of A satisfy

$$\begin{aligned}&|\tilde{\mu}_1|,|\tilde{\mu}_2|\asymp\delta,\\&|\tilde{\mu}_3|,|\tilde{\mu}_4|,|\tilde{\mu}_5|\asymp\delta^2\quad(0<\delta\ll 1),\end{aligned} \tag{5.47}$$

where $A\asymp B$ denotes $B/C\leqslant A\leqslant CB$ ($C>0$). So the eigenvalues μ of H_Λ satisfy

$$|\mu|\geqslant\mathcal{O}(1)\delta^2. \tag{5.48}$$

The eigenfunctions of H_Λ satisfy

$$\|\phi\|_{\ell^2(\partial\Lambda)}=\mathcal{O}(\delta^4) \tag{5.49}$$

by using the same arguments as in (5.6)–(5.11). (5.48), (5.49) imply that (H1), (H3) are satisfied and in fact $0\notin\sigma(H_\Lambda)$ for all finite subsets $\Lambda\subset\mathbb{Z}^2$. So we reach the conclusion of Theorem 1. $\square$

Proof of Theorem 2. If $u(0) \in H^s(\mathbb{T})$ $(s > 0)$, then $u(t) \in H^s(\mathbb{T})$ for all t and satisfies

$$\|u(t)\|_{H^s} \leqslant C_s\big(1 + |t|^s\big)\|u(0)\|_{H^s}. \tag{5.50}$$

This holds generally for linear Schrödinger equation of the form (1.1) with real, smooth and bounded V (depending on x and t), cf. [2, Lemma 6.2].

Let $\phi = \phi(j, n)$ be the eigenfunctions of the Floquet Hamiltonian H in (1.13) with eigenvalue E. Then the Bloch waves

$$\check{\phi}(x, t) = e^{iEt} \sum_{(j,n)\in\mathbb{Z}^2} \phi(j, n)e^{i(jx+nt)} \tag{5.51}$$

provide a basis to expand the solution $u(t)$ with initial condition $u(0)$. More precisely for any given initial condition $u(0) \in L^2(\mathbb{T})$, let $\hat{u}_0(j)$, $j \in \mathbb{Z}$ be its Fourier coefficients. Identifying $\hat{u}_0 \in \ell^2(\mathbb{Z})$ with $\tilde{u}_0 \in \ell^2(\mathbb{Z}^2)$ defined by

$$\begin{cases} \tilde{u}_0(j, 0) = \hat{u}_0(j), \\ \tilde{u}_0(j, n) = 0, \quad n \neq 0, \end{cases} \tag{5.52}$$

we have

$$u(x, t) = \sum_{\phi} (\tilde{u}_0, \phi)\check{\phi}(x, t). \tag{5.53}$$

So

$$\begin{aligned} \|u(t)\|^2_{H^s} &= \Big\|\sum_{\phi} (\tilde{u}_0, \phi)\check{\phi}(t)\Big\|^2_{H^s} \quad (s > 0) \\ &= \sum_j \big(1 + |j|^{2s}\big)\Big|\sum_{\phi} (\tilde{u}_0, \phi)\big(\check{\phi}(t), e_j\big)\Big|^2 \\ &= \sum_j \big(1 + |j|^{2s}\big)\Big|\sum_{\phi}\sum_k \hat{u}_0(k)\phi(k, 0)\big(\check{\phi}(t), e_j\big)\Big|^2, \end{aligned} \tag{5.54}$$

where

$$e_j = e^{-ijx}, \tag{5.55}$$

and

$$\big(\check{\phi}(t), e_j\big) = \int_0^{2\pi} \check{\phi}(t)e^{-ijx}\, dx = e^{iEt} \sum_{n\in\mathbb{Z}} \phi(j, n)e^{int}. \tag{5.56}$$

In view of the localization properties of ϕ in (1.14), (1.15), we decompose $\sum_k$ into $\sum_{k,||k|-|j||\leqslant|j|/2}$ and $\sum_{k,||k|-|j||>|j|/2}$. We have

$$\begin{aligned}\|u(t)\|_{H^s}^2 &= \sum_j \bigl(1+|j|^{2s}\bigr)\Bigl|\sum_\phi \sum_{k,||k|-|j||\leqslant |j|/2} \hat{u}_0(k)\phi(k,0)\bigl(\check{\phi}(t),e_j\bigr)\Bigr|^2 \\ &\quad + \sum_j \bigl(1+|j|^{2s}\bigr)\Bigl|\sum_\phi \sum_{k,||k|-|j||>|j|/2} \hat{u}_0(k)\phi(k,0)\bigl(\check{\phi}(t),e_j\bigr)\Bigr|^2 \\ &\stackrel{\text{def}}{=} S_1+S_2. \end{aligned} \tag{5.57}$$

Using (1.14), (1.15),

$$S_2 \leqslant C\sum_j \bigl(1+|j|^{2s}\bigr) \sum_{k,||k|-|j||>|j|/2} e^{-\alpha||j|-|k||}\bigl\|u(0)\bigr\|_{H^s} \tag{5.58}$$

for some $\alpha>0$ and we used $\|\hat{u}_0\|_\infty^2 \leqslant \|\hat{u}_0\|_2^2 \leqslant \|u(0)\|_{H^s}^2$ $(s>0)$. So

$$S_2 \leqslant C_s^{(1)}\bigl\|u(0)\bigr\|_{H^s}^2. \tag{5.59}$$

To estimate S_1, we notice that $||k|-|j||\leqslant |j|/2$. So

$$\frac{|j|}{2} \leqslant |k| \leqslant \frac{3}{2}|j| \leqslant 2|j|. \tag{5.60}$$

We make a dyadic expansion. Define

$$u_0^{(\ell)} = \sum_{\frac{1}{4}\cdot 2^\ell \leqslant |k| \leqslant 4\cdot 2^\ell} \hat{u}_0(k)e^{ikx}, \quad \ell=0,1,\ldots. \tag{5.61}$$

Then

$$\begin{aligned} S_1 &\leqslant C\sum_\ell \bigl(2^\ell\bigr)^{2s} \sum_{\frac{1}{2}\cdot 2^\ell \leqslant |j| \leqslant 2\cdot 2^\ell} \Bigl|\sum_\phi \sum_k \widehat{u_0^{(\ell)}}(k)\phi(k,0)\bigl(\check{\phi}(t),e_j\bigr)\Bigr|^2 \\ &\leqslant C\sum_\ell \bigl(2^\ell\bigr)^{2s} \Bigl\|\sum_k \sum_\phi \widehat{u_0^{(\ell)}}(k)\phi(k,0)\check{\phi}(t)\Bigr\|_2^2 \\ &= C\sum_\ell \bigl(2^\ell\bigr)^{2s} \Bigl\|\sum_\phi \bigl(u_0^{(\ell)},\phi\bigr)\check{\phi}(t)\Bigr\|_2^2 \\ &= C\sum_\ell \bigl(2^\ell\bigr)^{2s} \bigl\|u_0^{(\ell)}\bigr\|_2^2 \\ &\leqslant C_s^{(2)}\bigl\|u(0)\bigr\|_{H^s}^2, \end{aligned} \tag{5.62}$$

where we used (5.60), (5.61), ℓ^2 norm conservation. The last line follows by standard considerations using dyadic expansion. Using (5.59), (5.62) in (5.57), we obtain (1.16). □

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